Orinigal Article

$\delta\hat{}\ \theta_{\mathcal{X}}$ -Sets in Ideal Topological Spaces

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Abstract: Topology is a branch of mathematics concerned with generalization of the concepts of continuity. In 1847, Johann Benedict Listing introduce the term *topology*, although he had used term *Analysis Situs* for the past few years. Kuratowski and Vaidyanathaswamy introduced the subject of ideal in topological spaces for almost half a century ago, they motivated many researchers in applying topological ideals to generate other concepts and obtain properties that are analogous to the basic properties in general topology. In 1990, Jankovic and Hamlett obtained a new topology using old ones and introduced the notion of ideal topological spaces. They introduced the concept of I-open sets in ideal topological spaces in 1992, their work initialized the application of topological ideals in the generalization of most fundamental properties in general topology. This study investigated the concept of $\delta\hat{}\theta_{\mathcal{X}}$-sets in ideal topological spaces. Specifically, it aimed to: Define and establish some properties of $\delta\hat{}\theta_{\mathcal{X}}$-closed set and $\delta\hat{}\theta_{\mathcal{X}}$-open set in ideal topological spaces; and introduce and investigate its relationship to some other type of sets in ideal topological spaces. The researcher investigated the notion of $\delta\hat{}\theta_{\mathcal{X}}$-closed set in ideal topological space. It was proven that the countable union of $\delta\hat{}\theta_{\mathcal{X}}$-closed sets is $\delta\hat{}\theta_{\mathcal{X}}$-closed set. The complement of $\delta\hat{}\theta_{\mathcal{X}}$-closed set is known as $\delta\hat{}\theta_{\mathcal{X}}$-open set in $(X, \tau, \mathcal{X})$. The characterization of $\delta\hat{}\theta_{\mathcal{X}}$-open set was proven. Moreover, the researcher established the relationship of $\delta\hat{}\theta_{\mathcal{X}}$-closed sets to some other known type of closed sets in ideal topological space. Some of the basic properties of $\delta\hat{}\theta_{\mathcal{X}}$-closed and $\delta\hat{}\theta_{\mathcal{X}}$-open sets were also investigated in this paper.

Keywords: safety parameter, 3 MW TRIGA, JENDL, JEF

1. INTRODUCTION

Topology is a branch of mathematics concerned with generalization of the concepts of continuity [1]. In 1847, Johann Benedict Listing introduce the term *topology*, although he had used term *Analysis Situs* for the past few years. Krakowski and Vaidyanathaswamy introduced the subject of ideal in topological spaces for almost half a century ago, they motivated many researchers in applying topological ideals to generate other concepts and obtain properties that are analogous to the basic properties in general topology [2]. In 1990, Jankovic and Hamlett obtained a new topology using old ones and introduced the notion of ideal topological spaces [3]. They introduced the concept of I-open sets in ideal topological spaces in 1992, their work initialized the application of topological ideals in the generalization of most fundamental properties in general topology [4]. Levine introduced the concept of generalized closed sets briefly, g-closed sets and studied their most fundamental properties in topological spaces [5]. In 2008, Akdag introduced the notion of θ-I-open sets in ideal topo- logical space. Yuksel, Acikgos, and Noori define and introduce the concept of δ-I-closed sets and its complement [6]. This paper has great contribution in the field of mathematics particularly in topology by applying the notion of $\delta\hat{}\ \theta_{\mathcal{X}}$-open and $\delta\hat{}\ \theta_{\mathcal{X}}$-closed sets in ideal topological space. This paper can also be a good reading material and it can offer additional knowledge and information about $\hat{\delta}\theta_{\mathcal{X}}$-sets in ideal topological space. To the mathematics enthusiasts and researchers, they could use the results of this paper as a guide and can be a reference to their study in the field of topology [7]. In addition, recommendation is provided to give direction for future studies. This study focused on introducing a new class of sets in ideal topological space, namely $\hat{\delta}\theta_{\mathcal{X}}$-closed and open sets. It determined the relationship between these new classes of sets and some other known type of sets ideal topological space. The sets considered are the following; δ-I-closed sets, $\hat{\delta}$-closed sets, θ-I-closed sets, $\hat{\delta}$ s-closed sets and δ-I-open sets in ideal topo- logical space [8]. It also determined and established some basic properties of these new class of sets in ideal topological space such as containment and countable union of $\hat{\delta}\theta_{\mathcal{X}}$-closed sets. For further investigation, the researcher will be dealing the concept of σ-closure of a set, θ-I-closure of a set, and $\hat{\delta}\theta_{\mathcal{X}}$-closed sets. In this paper, the researcher introduced and investigated the notion of the new class of sets, namely $\hat{\delta}\theta_{\mathcal{X}}$-closed sets in ideal topological spaces. Moreover, the researcher investigated the relationship of $\hat{\delta}\theta_{\mathcal{X}}$-sets to some known class of sets in ideal topological spaces. This study investigated the concept of $\hat{\delta}\theta_{\mathcal{X}}$-sets in ideal topological spaces. Specifically, it aimed to: Define and establish some properties of $\hat{\delta}\theta_{\mathcal{X}}$-closed set and $\hat{\delta}\theta_{\mathcal{X}}$-open set in ideal topological spaces; and introduce and investigate its relationship to some other type of sets in ideal topological spaces.

2. MATERIALS AND METHODS

This presents the concepts that is needed in this study. Some examples are provided for clear understanding of the given concepts.

Definition 01 (*J. Dugundji 1975*) Let X be a nonempty set. A collection τ of subsets of X is a topology on X if it satisfies the following:

(i) $\emptyset, X \in \tau$

(ii) $\{M_\omega | \omega \in \Omega\} \subseteq \tau$ implies tω∈Ω $M_\omega \in \tau$

(iii) $A, B \in \tau$ implies that $A \cap B \in \tau$

If τ is a topology on X, then (X, τ) is called a topological space, and the elements of τ are called τ-open sets or simply open sets. A subset F of X is said to be τ-closed sets or simply closed sets if its complement X\F is open.

The closed subsets of a topological space satisfy the following properties:

(C_1) $\emptyset$ and X are closed sets

(C_2) Finite union of closed sets are closed

(C_3) Arbitrary intersections of closed sets are closed

Example 02 Consider the following classes of subsets of X = {a, b, c, d, e} τ_1 = {X, $\emptyset$, {a}, {c, d}, {a, c, d}, {b, c, d, e}}

τ_2 = {X, $\emptyset$, {a}, {c, d}, {a, c, d}, {b, c, d}, {a, b, d, e}}

Observe that τ_1 is a topology on X, since it satisfies the necessary axioms (i), (ii), and (iii). But τ_2 is not a topology on X since, the union of the set

{a, c, d} ∪ {b, c, d} = {a, b, c, d} does not belong to τ_2. Hence, τ_2 does not satisfy (ii).
Definition 03 The interior of A, denoted by int(A), is the union of all open sets contained in A. That is, int(A) = t {U ∈ τ: U ⊆ A}. Since the arbitrary union of open sets is open, we have the following remark.

Remark 04 Let (X, τ) be a topological space and A ⊆ X. Then,

(i) int(A) is open and,

(ii) int(A) is the largest open set contained in A.

Example 05 Consider the topology τ = {X, ∅, {a}, {c, d}, {a, c, d}, {b, c, d, e}} on X = {a, b, c, d, e}, and the subset of A = {b, c, d} of X. The point c and d are the interior points of A, since c, d ∈ {c, d} ⊂ A where {c, d} is an open set.

Definition 06 The closure of A, denoted by cl(A), is the intersection of all closed supersets of A. In other words, if {F_i: i ∈ I} is a collection of closed subsets of X containing A, then cl(A) = ui F_i.

Example 07 Consider the topology
τ = {X, ∅, {a}, {c, d}, {a, c, d}, {b, c, d, e}} on X = {a, b, c, d, e} where the closed
subsets of X are ∅, X, {b, c, d, e}, {a, b, e}, {b, e} and {a}. Suppose A = {b}, then the closure of A is cI(A) = cl({b}) = {b, c, d, e} ∩ {a, b, e} ∩ {b, e}. Therefore, the closure of A is {b, e}.

Definition 08 *(Kuratowski, 1996)* An ideal $\mathcal{X}$ on a set X is a nonempty collection of subsets of X which satisfies the following:

(i) A ∈ $\mathcal{X}$ and B ⊆ A implies B ∈ $\mathcal{X}$; and

(ii) A ∈ $\mathcal{X}$ and B ∈ $\mathcal{X}$ implies A ∪ B ∈ $\mathcal{X}$

Note that because of (i), ∅ is always a member of an ideal $\mathcal{X}$. In this context the symbol $\mathcal{X}$(X) is denoted an ideal $\mathcal{X}$ on a set X.

Definition 09 *(Kuratowski, 1996)* An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{X}$ on X and is denoted by (X, τ, $\mathcal{X}$).
Definition 10 *(Jankovic & Hamlett, 1990)* For every ideal topological
space (X, τ, $\mathcal{X}$), there exists a topology $\tau^{\wedge}$($\mathcal{X}$, τ), called the Λ-topology, generated by the base;
B ($\mathcal{X}$, τ) = {U\I|U ∈ τ and I ∈ $\mathcal{X}$}.

Note that the family $\tau^{\wedge}$(B ($\mathcal{X}$, τ) that consists of ∅, X, and all unions of members of B ($\mathcal{X}$, τ) is a topology on X.

Example 11 Let X = {a, b, c, d},
τ = {X, ∅, {a}, {c}, {a, c}, {c, d}, {a, c, d}, {b, c, d}}, $\mathcal{X}$ = {∅. {a}} and let U ∈ τ. From Definition 10, U ∈ τ and ∅ ∈ $\mathcal{X}$ implies U\∅ ∈ B ($\mathcal{X}$, τ). Also, U ∈ τ and {a} ∈ $\mathcal{X}$ implies U\{a} ∈ B ($\mathcal{X}$, τ). Note that U\∅ are the sets X, ∅, {a}, {c}, {a, c}, {c, d}, {a, c, d}, {b, c, d}; and U\{a} are the sets ∅, {c}, {c, d} and {b, c, d}. Hence, B ($\mathcal{X}$, τ) = {X, ∅, {a}, {c}, {a, c}, {c, d}, {a, c, d}, {b, c, d}}, and so $\tau^{\wedge}$(B ($\mathcal{X}$, τ) = {X, ∅, {a}, {c}, {a, c}, {c, d}, {a, c, d}, {b, c, d}},

For brevity the researcher writes $\tau^{\wedge}$ for $\tau^{\wedge}$($\mathcal{X}$, τ) or $\tau^{\wedge}$(B ($\mathcal{X}$, τ)) and denote the interior and closure of A in (X, $\tau^{\wedge}$) as int^(A) and cl^(A), respectively.

Definition 12 *(Velichko, 1968)* A subset A of X in ideal topological space (X, τ, $\mathcal{X}$) is called θ-I-closed set if A = cl^(A), where
cl^(A) = {x ∈ X: cl^(U) $\cap_\theta$ A /= ∅, for each U ∈ τ and x ∈ U}. θ

The complement of θ-I-closed set is called θ-I-open set.

Example 13 Consider the ideal topological space in Example 11. Note that
$\tau^{\wedge}$ = {X, ∅, {a}, {c}, {a, c}, {c, d}, {a, c, d}, {b, c, d}}
$\tau^{\wedge}$-closed= {∅, X, {b, c, d}, {a, b, d}, {b, d}, {a, b}, {b}, {a}}.

Suppose A = {a} then for x = {a} and U({a}) = {a}, {a, c}, {a, c, d}and X. It follows that, cl^({a}) ∩{a} = {a}, cl^({a, c}) ∩{a} = {a}, cl^({a, c, d}) ∩{a} = {a}, cl^(X) ∩ {a} = {a}.

Since {a} satisfies the condition. Hence, it is a member of cl^(A). Now, for x = {b}, U({b}) = {b, c, d} and X. It follows that, cl^({b, c, d}) ∩ {a} = ∅.
θ

Since {b} does not satisfy the condition. Hence, it is not a member of cl^(A). For x = {c}, U({c}) = {c}, {a, c}, {c, d}, {a, c, d}, {b, c, d} and X. Now, cl^({c}) ∩ {a} = ∅.

Since one member of U({c}) does not satisfy the condition, so {c} is not a member of cl^(A). For x = {d}, U({d}) = {c, d}, {a, c, d}, {b, c, d} and X. It follows that, cl^({c, d}) $\cap_\theta$ {a} = ∅.

Hence, {d} is not a member of cl^(A). Thus, cl^(A) = cl^({a}) = {a} = A.
θ θ θ

Therefore, A = {a} is θ-I-closed sets. Similarly, X, ∅, {a} and {b, c, d} are the θ-I-closed sets and ∅, X, {b, c, d} and {a} are the θ-I-open sets.

In 2008, Akdag established the notion of θ-I-open sets and θ-closed sets. He found out that θ-I-open satisfy the arbitrary union and finite intersection and established a theorem that $\tau_\theta \subseteq \tau_{\theta-I} \subseteq \tau \subseteq \tau^{\wedge}$ and so it has a following remark:

Remark 14 Since $\tau_\theta \subseteq \tau_{\theta-I} \subseteq \tau$, the following are true;

(i) $int_\theta(A)$ ⊆ int^(A) ⊆ int_θ(A) ⊆ int^(A) ⊆ A

(ii) A ⊆ cl^(A) ⊆ cl(A) ⊆ cl^(A) ⊆ $cl_\theta(A)$ θ

On 2015, Navaneethakrishnan et al. introduced the notion of δˆ-closed sets. Some of its known results that can be used in the development of this paper are the following:

Definition 15 A subset A of an ideal topological space (X, τ, $\mathcal{X}$) is called δˆ-closed if σcl(A) ⊂ U whenever A ⊂ U and U is open set.

Theorem 16 *Every* θ-*I-closed set is* δˆ*-closed set.*

Theorem 17 *Let* (X, τ, $\mathcal{X}$) *be an ideal topological space and* A ⊆ X. *If* A ⊆ B ⊆ σ*cl*(A) *then,* σ*cl*(A) = σ*cl*(B).

Yuksel et al. established the concept of δ-I-closed set that if σcl(A) = A then A is δ-I-closed set in (X, τ, $\mathcal{X}$). Moreover, they obtained the following;

Lemma 18 *Let* A *and* B *be subsets of an ideal topological space* (X, τ, $\mathcal{X}$). *Then, the following properties hold.*

(i) A ⊆ σ*cl*(A)

(ii) *If* A ⊂ B*, then* σ*cl*(A) ⊂ σ*cl*(B)
(iii) σ*cl*(A) = ∪ {F ⊂ X | A ⊂ F *and* F *is* δ*-I-closed*}
(iv) *If* A *is* δ*-I-closed set of X for each* α ∈ Δ*, then* ∪ {A_α | α ∈ Δ} *is* δ*-I-closed*

(v) σ*cl*(A) *is* δ*-I-closed.*

The main purpose of this study is to define and introduce the new class of sets which are δˆ $\theta_{\mathcal{X}}$-closed sets and δˆ $\theta_{\mathcal{X}}$-open sets in ideal topological space. First the researcher defined δˆ $\theta_{\mathcal{X}}$-closed set and δˆ $\theta_{\mathcal{X}}$-open set in ideal topological space by applying the concept of σ-closure and provide examples on it. After that, the researcher studied and established some of the basic properties of δˆ $\theta_{\mathcal{X}}$-closed set and δˆ $\theta_{\mathcal{X}}$-open set in ideal topological space. Lastly δˆ $\theta_{\mathcal{X}}$-closed set and δˆ $\theta_{\mathcal{X}}$-open set are being compared to some other known type of sets in ideal topological spaces namely; δ-I-closed sets, δˆ-closed sets, θ-I-closed sets, δˆ s-closed sets, δ-I-open sets in (X, τ, $\mathcal{X}$) and established theorems that δˆ $\theta_{\mathcal{X}}$-closed set is bigger than those being compared. The researcher established some basic properties of δˆ $\theta_{\mathcal{X}}$-closed set in (X, τ, $\mathcal{X}$) and investigated the relationship of δˆ $\theta_{\mathcal{X}}$-closed set in (X, τ, $\mathcal{X}$) to some other known type of closed sets in (X, τ, $\mathcal{X}$).

δˆ $\theta_{\mathcal{X}}$ -Closed Set δˆ θ_I-closed set if

Definition 19 A subset A of X in (X, τ, $\mathcal{X}$) is called a σcl(A) ⊆ U, whenever A ⊆ U and U is θ-I-open.

Example 20 Consider the ideal topological space in Example 11. Note that, τ = {X, ∅, {a}, {c}, {a, c}, {c, d}, {a, c, d}, {b, c, d}}. Now, τ-closed are X, ∅,

{b, c, d}, {a, b, d}, {b, d}, {a, b}, {b} and {a}. From Example 13,

$\tau^{\wedge}$ = {X, ∅, {a}, {c}, {a, c}, {c, d}, {a, c, d}, {b, c, d}}

and θ-I-open sets are X, ∅, {b, c, d} and {a}. Then $\tau^{\wedge}$-closed are X, ∅, {b, c, d}

, {a, b, d}, {b, d}, {a, b}, {b}, {a} and θ-I-closed sets are X, ∅, {a} and {b, c, d}. Suppose A = {a}, then σcl(A) = {x ∈ X:int ($cl^{\wedge}$(U)) ∩ A /= ∅, for each open set U(x)} that is for x = {a}, U({a}) = {a}, {a, c}, {a, c, d} and X.

Now, int($cl^{\wedge}$({a})) ∩ {a} = {a} ∩ {a} = {a} =/ ∅,

Int ($cl^{\wedge}$({a, c})) ∩ {a} = X ∩ {a} = {a} =/

Int ($cl^{\wedge}$({a, c, d})) ∩{a} = X∩{a} = {a} =/

∅, int($cl^{\wedge}$(X)) ∩{a} = X∩{a} = {a} =/ ∅.

Since {a} satisfies the condition to all U(x), then {a} is a member of σcl(A).

For x = {b}, U({b}) = {b, c, d} and X. Now,

Int (cl∧({b, c, d})) ∩ {a} = {b, c, d} ∩ {a} = ∅. Since {b} does not satisfy the condition. Hence, {b} is not a member of σcl(A). For x = {c},

U({c}) = {c}, {a, c}, {a, c, d}, {b, c, d} and X. Now,

int(cl∧({c})) ∩ {a} = {b, c, d} ∩ {a} = ∅. Since {c} does not meet the condition, it means that {c} is not a member of σcl(A). For x = {d},

U({d}) = {c, d}, {a, c, d}, {b, c, d} and X. Now,

Int (cl∧({c, d})) ∩ {a} = {b, c, d} ∩ {a} = ∅. Since {d} does not satisfy the necessary condition, so {d} is not a σcl(A). Hence, for A = {a} the σcl(A) = {a}. It follows that σcl(A) = σcl({a}) = {a} ⊆ {a}, {a, c}, X. Therefore, {a} is

δˆ θI-closed. Similarly, δˆ θ𝓧-closed sets are X, ∅, {a}, {b}, {d}, {a, b}, {a, d}, {b, c},

{b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d} and {b, c, d}.

Theorem 21 Let (X, τ, 𝓧) be an ideal topological space. If A is δˆ θ𝓧 -closed subset of X and A ⊆ B ⊆ σcl(A), then B is also δˆ θ𝓧 -closed set in (X, τ, 𝓧).

Proof. Let A be δˆ θ𝓧-closed set in (X, τ, 𝓧) and U be θ-I-open set such B ⊆ U. Since A is δˆ θ𝓧-closed set, then σcl(A) ⊆ U and A ⊆ U. Now, by Theorem 2.17, A ⊆ B ⊆ σcl(A) implies σcl(A) = σcl(B). Thus, σcl(B) ⊆ U. Therefore, B is δˆ θ𝓧-closed set in (X, τ, 𝓧). □

Remark 22 The converse of the above theorem need not be true as shown in the following example.

Example 23 Let X = {a, b, c, d}, A = {a, c}, B = {a, c, d} then

Σcl ({a, c}) = X. Now, {a, c} ⊆ {a, c, d} ⊆ X. But {a, c} is not δ̂ θ𝓧-closed set.

Theorem 24 *let* (X, τ, 𝓧) *be an ideal topological space. If* A *is* $\hat{\delta}\theta_{\mathcal{X}}$ *-closed set in* (X, τ, 𝓧)*, then* σ*cl*(A) − A *does not contain any non-empty* θ*-I-closed set in* (X, τ, 𝓧).

Proof. Let A be $\hat{\delta}\theta_{\mathcal{X}}$-closed set in (X, τ, 𝓧). Suppose F be a nonempty θ-I- closed set such that F ⊆ σcl(A)−A. Then, F ⊆ σcl(A)∩ (X −A) which implies F ⊆ σcl(A) and F ⊆ X−A. By definition of $\hat{\delta}\theta_{\mathcal{X}}$-closed set σcl(A) ⊆ X−F such that A ⊆ X − F where X − F is θ-I-open set. It implies that F ⊆ X − σcl(A). Since F ⊆ σcl(A) and F ⊆ X − σcl(A), it follows that

F ⊆ σcl(A) ∩ (X − σcl(A)) = ∅. Thus F = ∅, which is a contradiction. Therefore, σcl(A) − A does not contain any nonempty θ-I-closed set in (X, τ, 𝓧). □

Remark 25 The inverse of the above theorem is not always true as shown in the following example.

Example 26 Let X = {a, b, c, d}, A = {c}, then σcl(A) −A = {b, c, d} −{c} = {b, d}, does not contain any non-empty θ-I-closed. But A is not $\hat{\delta}\theta_{\mathcal{X}}$-closed set.

Theorem 27 *Let* (X, τ, 𝓧) *be an ideal topological space. If* {A_i: i ∈ I} *be a collection of* θ*-I-open sets in* (X, τ, 𝓧)*, then* t A_i *is* θ*-I-open set in* (X, τ, 𝓧).i∈I

Proof. Let {A_i: i ∈ I} be a family of θ-I-open sets in (X, τ, 𝓧). By definition of

θ-I-open set int^(A_i) = A_i, hence int^ (tA_i) = t int^(A_i) = t A_i. Therefore,

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i∈I i∈I

t A_i is θ-I-open set in (X, τ, 𝓧). □

i∈I

Theorem 28 *Let* (X, τ, 𝓧) *be an ideal topological space. If* {A_i: i ∈ I} *is a collection of* $\hat{\delta}\theta_{\mathcal{X}}$ *-closed sets in* (X, τ, 𝓧)*, then* t A_i *is* $\hat{\delta}\theta_{\mathcal{X}}$ *-closed set in* (X, τ, 𝓧).

i∈I

Proof. Let A_i be $\hat{\delta}\theta_{\mathcal{X}}$-closed set for all i ∈ I. Then σcl (A_i) ⊆ U_i where U_i is any θ-I-open set such that A_i ⊆ U_i for each i ∈ I. Note that, A_i ⊆ U_i implies t A_i ⊆ t U_i and by Theorem 3.9, t U_i is θ-I-open. Now,

i∈I i∈I i∈I

σcl (tA_i) ⊆ t σcl(A_i) ⊆ t U_i. Therefore, t A_i is δ̂ θ − 𝓧-closed set in

Remark 29 The finite intersection of δˆ θ$_{\mathcal{X}}$-closed set in (X, τ, $\mathcal{X}$) is not always δˆ θ$_{\mathcal{X}}$-closed set in (X, τ, $\mathcal{X}$) as shown in the following example.

Example 30 From Example 41 the set {a, b, c} and {a, c, d} are both

δˆθ$_{\mathcal{X}}$-closed set but {a, c, d} ∩ {a, b, c} = {a, c} is not δˆ θ$_{\mathcal{X}}$-closed set.

Theorem 31 Let (X, τ, $\mathcal{X}$) be an ideal topological space. Then every

δ-I-closed set in (X, τ, $\mathcal{X}$) is δˆ θI-closed set in (X, τ, $\mathcal{X}$).

Proof. Let A be δ-I-closed set and U be any θ-I-open set containing A. Since A is δ-I-closed, σcl(A) = A for every subset A of X. It follows that σcl(A) ⊆ U. Hence, A is δˆ θI-closed set in (X, τ, $\mathcal{X}$). □

Remark 32 The converse of the above theorem need not be true as shown in the following example.

Example 33 From Example 20, δ-I-closed are X, ∅, {a}, {b, d}, {a, b, d},

{b, c, d}. Now, {a, c, d} is δˆ θI-closed set but not δ-I-closed.

Theorem 34 Let (X, τ, $\mathcal{X}$) be an ideal topological space. Then σcl(A) is δˆ θ$\mathcal{X}$ -closed set in (X, τ, $\mathcal{X}$).

Proof. Let U be any θ-I-open set containing σcl (A) such that σcl(A) ⊆ U. By Lemma 18 (v) σcl(A) is δ-I-closed in (X, τ, $\mathcal{X}$). And so, by Theorem 31, σcl(A) is δˆθ$\mathcal{X}$-closed set in (X, τ, $\mathcal{X}$). □

Theorem 35 Every θ-I-open set in (X, τ, $\mathcal{X}$) is open in (X, τ).

Proof. Let A be an θ-I-open set in (X, τ, $\mathcal{X}$). Then by definition of θ-I-open, int∧(A) = A. It follows that, A = int∧(A) ⊆ int(A) by Remark 14. Note that

θ θ

int(A) ⊆ A, and so int(A) = A. Thus, A is open set in (X, τ). □

Theorem 36 Let (X, τ, $\mathcal{X}$) be an ideal topological space. Then every

δˆ-closed set in (X, τ, $\mathcal{X}$) is δˆ θI-closed set in (X, τ, $\mathcal{X}$).

Proof. Let A be a δˆ-closed set in (X, τ, $\mathcal{X}$) and U be any θ-I-open set in (X, τ, $\mathcal{X}$) such that A ⊆ U. Then, by Theorem 35, U is open in (X, τ) such that A ⊆ U. Since A is δˆ-closed set, σcl(A) ⊆ U whenever A ⊆ U and U is open set. It follows that A is δˆ θI-closed set in (X, τ, $\mathcal{X}$). □

Remark 37 The converse of Theorem 36 is not always true as shown in the following example.

Example 38 From Example 20, δ^-closed sets are X, ∅, {a}, {b}, {d}, {a, b},

{b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}. The set {a, d} is δ^ θI-closed but not

δ^-closed set in (X, τ, $\mathcal{X}$).

Theorem 39 *Let* (X, τ, $\mathcal{X}$) *be an ideal topological space. Then every*

θ-*I-closed set in* (X, τ, $\mathcal{X}$) *is* δ^ θ_I-*closed set in* (X, τ, $\mathcal{X}$).

Proof. Let A be θ-I-closed set in (X, τ, $\mathcal{X}$). By Theorem 16 A is δ^-closed set in (X, τ, $\mathcal{X}$). Then by Theorem 36 A is also δ^ θ_I-closed set in (X, τ, $\mathcal{X}$). □

Remark 40 The converse of Theorem 39 need not be true as shown in the following example.

Example 41 From the Example 20 δ^ θ_I-closed sets is X, ∅, {a}, {b}, {d},

{a, b}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}. Now, the set

{a, b, c} is δ^ θ_I-closed but not θ-I-closed set.

Theorem 42 *Every* θ-*I-open set in* (X, τ, $\mathcal{X}$) *is semi-open in* (X, τ).

Proof. Let A be an θ-I-open set in (X, τ, $\mathcal{X}$). Then, by definition of θ-I-open, A = $int^{\wedge}$(A).

It follows that A = $int^{\wedge}$ ⊆ int(A) ⊆ cl(A) ⊆ cl(int(A)) by Remark

θ θ

2.14. Since A is θ-I-open, A is open in (X, τ) by Theorem 35 and so

A = int(A). Thus, A ⊆ int(cl(A)). Hence, A is semi-open set in (X, τ). □

Theorem 43 *Let* (X, τ, $\mathcal{X}$) *be an ideal topological space. Then every*

δ^ s-*closed set in* (X, τ, $\mathcal{X}$) *is* δ^ θ_I -*closed set in* (X, τ, $\mathcal{X}$).

Proof. Let A be a δ^ s-closed set and U be any θ-I-open set containing A. By Theorem 42, U is also a semi-open in (X, τ, $\mathcal{X}$) such that A ⊆ U. Since A is δ^s-closed set, σcl (A) ⊆ U. It follows that A is δ^ $\theta_{\mathcal{X}}$-closed set.

Remark 44 The converse of the above theorem need not be true as shown in the following example.

Example 45 From the Example 20, δ^ s-closed set are

X, ∅, {a}, {b, d}, {a, b, d}, {b, c, d}. The set {b, c} is δ^ $\theta_{\mathcal{X}}$-closed but not δ^ s-closed.

This presents some basic properties of $\hat{\delta}\ \theta_{\mathcal{X}}$-open set in (X, τ, $\mathcal{X}$) and established the relationship with some other known type of open set in (X, τ, $\mathcal{X}$).

Definition 46 Let (X, τ, $\mathcal{X}$) be an ideal topological space and A ⊆ X. Then

A is called $\hat{\delta}\ \theta_{\mathcal{X}}$-open set if its complement is $\hat{\delta}\ \theta_{\mathcal{X}}$-closed set.

Example 47 From Example 41, The set {a} is $\hat{\delta}\ \theta_{\mathcal{X}}$-closed. And so

$\{a\}^c$ = {b, c, d}. Thus {b, c, d} is $\hat{\delta}\ \theta_{\mathcal{X}}$-open set. The complement of $\hat{\delta}\ \theta_{\mathcal{X}}$-closed in Example 20 is $\hat{\delta}\ \theta_{\mathcal{X}}$-open which are ∅, X, {b, c, d}, {a, c, d}, {a, b, c}, {c, d}, {b, c},

{a, d}, {a, c}, {a, b}, {d}, {c}, {b} and {a}.

Theorem 48 *Let* (X, τ, $\mathcal{X}$) *be an ideal topological space. A subset* A *of* X *is*

$\hat{\delta}\ \theta_{\mathcal{X}}$ *-open set if and only if* U ⊆ σ*int*(A)*, whenever* U ⊆ A *and* U *is* θ-$\mathcal{X}$*-closed.*

Proof. Let A be a $\hat{\delta}\ \theta_{\mathcal{X}}$-open set in (X, τ, $\mathcal{X}$) and U be θ-I-closed where U ⊆ A. Note that X − A is $\hat{\delta}\ \theta_{\mathcal{X}}$-closed and X − U is θ-I-open where X − A ⊆ X − U. By Definition 19, σcl (X − A) ⊆ X − U and so X − σint(A) ⊆ X − U which implies U ⊆ σint(A). Conversely, suppose U ⊆ σint(A) where U ⊆ A and U is θ-I-closed. Then, X − σint(A) ⊆ X − U and it implies that σcl (X − A) ⊆ X − U. Note that X − U is θ-I-open and X − A ⊆ X − U. Hence, X − A is $\hat{\delta}\ \theta_{\mathcal{X}}$-closed set and so A is $\hat{\delta}\ \theta_{\mathcal{X}}$-open set. □

Theorem 49 *Let* (X, τ, $\mathcal{X}$) *be an ideal space and* A ⊆ X*. If* A *is* $\hat{\delta}\ \theta_{\mathcal{X}}$ *-open and* σ*int*(A) ⊆ B ⊆ A*, then* B *is* $\hat{\delta}\ \theta_{\mathcal{X}}$ *-open set in* (X, τ, $\mathcal{X}$).

Proof. Let A be $\hat{\delta}\ \theta_{\mathcal{X}}$-open set in (X, τ, $\mathcal{X}$) and U be θ-I-closed set such that

U ⊆ B. Since A is $\hat{\delta}\ \theta_{\mathcal{X}}$-open set, then U ⊆ σint(A) and U ⊆ A. Note

that σint(A) ⊆ B ⊆ A implies σcl (A^c) ⊇ B^c ⊇ A^c and so by Theorem 2.17, σcl (A^c) = σcl (B^c). It follows that σint(A) = σint(B) and hence σint(B) ⊇ U. Therefore, B is $\hat{\delta}\ \theta_{\mathcal{X}}$ in (X, τ, $\mathcal{X}$). □

Theorem 50 *Let* (X, τ, $\mathcal{X}$) *be an ideal topological space. If* A_i *is* θ*-I-closed for each* i ∈ I*, then* u A_i *is* θ*-I-closed set in* (X, τ, $\mathcal{X}$) i∈I

Proof. Let {A_i: i ∈ I} be a family of θ-I-closed sets in (X, τ, $\mathcal{X}$). By definition

of θ-I-closed set A_i = cl^(A_i), hence u A_i = u cl^(A_i) = cl^ (uA_i). Therefore,

θ i∈I θi ∈I θi∈I

u A_i is θ-I-closed set in (X, τ, $\mathcal{X}$). i∈I

Theorem 51 *Let* (X, τ, $\mathcal{X}$) *be an ideal topological space. If* {A_i: i ∈ I} *is a collection of* δˆ $\theta_{\mathcal{X}}$ *-open set in* (X, τ, $\mathcal{X}$)*, then* ∪ A_i *is* δˆ $\theta_{\mathcal{X}}$ *-open set in* (X, τ, $\mathcal{X}$)*.* i∈I

Proof. Let A_i be δˆ $\theta_{\mathcal{X}}$-open set for each i ∈ I. Then U_i ⊆ σint (A_i) where

U_i is any θ-I-closed set such that $U_i \subseteq A_i$ for each i ∈ I. It follows that

∪ U_i ⊆ ∪ σint (A_i) and ∪ U_i ⊆ ∪ A_i where ∪ U_i is θ-I-closed by Theorem

i∈I i∈I i∈I i∈I i∈

Remark 52 The arbitrary union of δˆ θ$\mathcal{X}$-open set in (X, τ, $\mathcal{X}$) is not always

δˆ θ$\mathcal{X}$-open set in (X, τ, $\mathcal{X}$) as shown in the following example.

Example 53 The complement of δˆ θ$\mathcal{X}$-closed set an Example 41 are δˆ θ$\mathcal{X}$-open sets. The set {b} and {d} are δˆ θ$\mathcal{X}$-open sets but {b} ∪ {d} = {b, d} is not δˆ θ$\mathcal{X}$-open set.

Theorem 54 Let (X, τ, $\mathcal{X}$) be an ideal topological space. Then every δ-I-open set in (X, τ, $\mathcal{X}$) is δˆ θ$\mathcal{X}$ -open set in (X, τ, $\mathcal{X}$).

Proof. Let A be δ-I-open set and U be any θ-I-closed set such that U ⊆ A. Since A is δ-I-open by its definition, A = σint(A) for every subset A of X. It follows that U ⊆ σint(A). Hence, A is δˆ θ$\mathcal{X}$-open set in (X, τ, $\mathcal{X}$). □

Remark 55 The converse of the above theorem need not be true as shown in the following example.

Example 56 Consider the δˆ θ$\mathcal{X}$-open set an Example 3.29. The set {a, b, c} is δˆ θ$\mathcal{X}$-open but not δ-I-open.

3. RESULT & DISCUSSION

The following are the results obtained by the researcher in this study.

(**Definition 19**) A subset A of X in (X, τ, $\mathcal{X}$) is called a δˆ θ_I-closed set if σcl (A) ⊆ U, whenever A ⊆ U and U is θ-I-open.

(**Theorem 21**) Let (X, τ, $\mathcal{X}$) be an ideal topological space. If A is δˆ $\theta_{\mathcal{X}}$-closed subset of X and A ⊆ B ⊆ σcl(A), then B is also δˆ $\theta_{\mathcal{X}}$-closed set in (X, τ, $\mathcal{X}$).

(**Remark 22**) The converse of the above theorem need not be true.

(**Theorem 24**) let (X, τ, $\mathcal{X}$) be an ideal topological space. If A is δˆ θ$_{\mathcal{X}}$-closed set in (X, τ, $\mathcal{X}$), then σcl (A) − A does not contain any non-empty θ-I-closed set in (X, τ, $\mathcal{X}$).

(**Remark 25**) The inverse of the above theorem is not always true.

(**Theorem 27**) Let (X, τ, $\mathcal{X}$) be an ideal topological space. If {A_i: i ∈ I} be a collection of θ-I-open set in (X, τ, $\mathcal{X}$), then t A_i is θ-I-open set in i∈I (X, τ, $\mathcal{X}$).

(**Theorem 28**) Let (X, τ, $\mathcal{X}$) be an ideal topological space. If {A_i: i ∈ I} is a collection of δˆ θ$_{\mathcal{X}}$-closed sets in (X, τ, $\mathcal{X}$), then t A_i is δˆ θ$_{\mathcal{X}}$-closed set i∈I in (X, τ, $\mathcal{X}$).

(**Remark 29**) The finite intersection of δˆ θ$_{\mathcal{X}}$-closed set in (X, τ, $\mathcal{X}$) is not always δˆ θ$_{\mathcal{X}}$-closed set in (X, τ, $\mathcal{X}$).

(**Theorem 31**) Let (X, τ, $\mathcal{X}$) be an ideal topological space. Then every δ-I-closed set in (X, τ, $\mathcal{X}$) is δˆ θ_I-closed set in (X, τ, $\mathcal{X}$).

(**Remark 32**) The converse of the above theorem need not be true.

(**Theorem 34**) Let (X, τ, $\mathcal{X}$) be an ideal topological space. Then σcl (A) is δˆ θ$_{\mathcal{X}}$-closed set in (X, τ, $\mathcal{X}$).

(**Theorem 35**) Every θ-I-open set in (X, τ, $\mathcal{X}$) is open in (X, τ).

(**Theorem 36**) Let (X, τ, $\mathcal{X}$) be an ideal topological space. Then every δˆ-closed set in (X, τ, $\mathcal{X}$) is δˆ θ_I-closed set in (X, τ, $\mathcal{X}$).

(**Remark 37**) The converse of Theorem 3.18 is not always true.

(**Theorem 39**) Let (X, τ, $\mathcal{X}$) be an ideal topological space. Then every θ-I-closed set in (X, τ, $\mathcal{X}$) is δˆ θ_I-closed set in (X, τ, $\mathcal{X}$).

(**Remark 40**) The converse of Theorem 3.21 need not be true.

(**Theorem 42**) Every θ-I-open set in (X, τ, $\mathcal{X}$) is semi-open in (X, τ).

(**Theorem 43**) Let (X, τ, $\mathcal{X}$) be an ideal topological space. Then every δˆ s-closed set in (X, τ, $\mathcal{X}$) is δˆ θ_I-closed set in (X, τ, $\mathcal{X}$).

(**Remark 44**) The converse of the above theorem need not be true.

(**Definition 46**) Let (X, τ, $\mathcal{X}$) be an ideal topological space and A ⊆ X. Then A is called δˆ θ$_{\mathcal{X}}$-open set if its complement is δˆ θ$_{\mathcal{X}}$-closed set.

(**Theorem 48**) Let (X, τ, $\mathcal{X}$) be an ideal topological space. A subset A of X is δˆ θ$_{\mathcal{X}}$-open set if and only if U ⊆ σint (A), whenever U ⊆ A and U is θ-$\mathcal{X}$-closed.

(**Theorem 49**) Let (X, τ, $\mathcal{X}$) be an ideal space and A ⊆ X. If A is δˆ θ$_{\mathcal{X}}$-open and σint(A) ⊆ B ⊆ A, then B is δˆ θ$_{\mathcal{X}}$-open set in (X, τ, $\mathcal{X}$).

(**Theorem 50**) Let (X, τ, $\mathcal{X}$) be an ideal topological space. If A_i is θ-I-closed for each i ∈ I, then u A_i is θ-I-closed set in (X, τ, $\mathcal{X}$) i∈I

(**Theorem 51**) Let (X, τ, $\mathcal{X}$) be an ideal topological space. If {A_i: i ∈ I} is a collection of δˆ θ$_{\mathcal{X}}$-open set in (X, τ, $\mathcal{X}$), then u A_i is δˆ θ$_{\mathcal{X}}$-open set in i∈I (X, τ, $\mathcal{X}$).

(**Remark 52**) The arbitrary union of δˆ θ$_{\mathcal{X}}$-open set in (X, τ, $\mathcal{X}$) is not always δˆ θ$_{\mathcal{X}}$-open set in (X, τ, $\mathcal{X}$).

(**Theorem 54**) Let (X, τ, $\mathcal{X}$) be an ideal topological space. Then every δ-I-open set in (X, τ, $\mathcal{X}$) is δˆ θ$_{\mathcal{X}}$-open set in (X, τ, $\mathcal{X}$).

(**Remark 55**) The converse of the above theorem need not be true.

4. CONCLUSION

The researcher investigated the notion of δˆ θ$_{\mathcal{X}}$-closed set in ideal topological space. It was proven that the countable union of δˆθ$_{\mathcal{X}}$-closed sets is δˆθ$_{\mathcal{X}}$-closed set. The complement of δˆθ$_{\mathcal{X}}$-closed set is known as δˆθ$_{\mathcal{X}}$-open set in (X, τ, $\mathcal{X}$). The characterization of δˆθ$_{\mathcal{X}}$-open set was proven. Moreover, the researcher established the relationship of δˆθ$_{\mathcal{X}}$-closed sets to some other known type of closed sets in ideal topological space. Some of the basic properties of δˆθ$_{\mathcal{X}}$-closed and δˆθ$_{\mathcal{X}}$-open sets were also investigated in this paper.

5. RECOMMENDATIONS

Investigate δˆ θ$_{\mathcal{X}}$-connectedness, compactness and separation of axioms, Introduce and investigate δˆ θ$_{\mathcal{X}}$-sets in other type of spaces namely; ideal ditopological space, ideal supra topological space and fuzzy ideal topological space.

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